

MODULI OF SHEAVES SUPPORTED ON CURVES OF GENUS TWO IN A QUADRIC SURFACE

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ABSTRACT. We study the moduli space of stable sheaves of Euler characteristic 1, supported on curves of arithmetic genus 2 contained in a smooth quadric surface. We show that this moduli space is rational. We give a classification of the stable sheaves involving locally free resolutions or extensions. We compute the Betti numbers by studying the variation of the moduli spaces of α -semi-stable pairs.

1. INTRODUCTION

Let \mathbb{P}^1 be the complex projective line and let \mathcal{F} be a coherent algebraic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with support of dimension 1. We fix the polarization $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. According to [1, Proposition 2], there are $r, s, t \in \mathbb{Z}$ such that for any $m, n \in \mathbb{Z}$ the Euler characteristic of the twisted sheaf $\mathcal{F}(m, n)$ satisfies $\chi(\mathcal{F}(m, n)) = rm + sn + t$. The linear polynomial $P_{\mathcal{F}}(m, n) = rm + sn + t$ is called the *Hilbert polynomial* of \mathcal{F} and the ratio $p(\mathcal{F}) = t/(r + s)$ is called the *slope* of \mathcal{F} with respect to the fixed polarization. We recall that \mathcal{F} is *semi-stable* (respectively *stable*) with respect to the above polarization if it does not contain subsheaves with support of dimension zero and for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $p(\mathcal{E}) \leq p(\mathcal{F})$ (respectively $p(\mathcal{E}) < p(\mathcal{F})$). According to [19], for a given polynomial P , there is a coarse moduli space, denoted $M(P)$, that is a projective variety, and that parametrizes S-equivalence classes of semi-stable sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial P . Its dimension, as computed in [13, Proposition 2.3], is $2rs + 1$. By the argument at [13, Theorem 3.1] $M(P)$ is irreducible and by [13, Proposition 2.3] it is smooth at the points given by stable sheaves.

The first non-trivial examples of such moduli spaces are $M(2m + 2n + 1)$ and $M(2m + 2n + 2)$. They were studied in [1] which contains a classification of the semi-stable sheaves by means of locally free resolutions. The rationality of $M(2m + 2n + 2)$ was proved in [5] by the wall-crossing method and in [17] by an elementary method.

The object of this paper is the study of $\mathbf{M} = M(3m + 2n + 1)$. The points of \mathbf{M} are stable sheaves \mathcal{F} supported on curves of bidegree $(2, 3)$ contained in $\mathbb{P}^1 \times \mathbb{P}^1$, with $\chi(\mathcal{F}) = 1$. As noted above, \mathbf{M} is a smooth projective variety of dimension 13. Twisting by powers of the polarization provides isomorphisms $\mathbf{M} \simeq M(3m + 2n + 5t)$ for any $t \in \mathbb{Z}$.

For $i, j \in \mathbb{Z}$ we use the abbreviation $\mathcal{O}(i, j) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(i, j)$. We fix vector spaces V_1 and V_2 over \mathbb{C} of dimension 2 and we make the identifications

$$\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V_1) \times \mathbb{P}(V_2), \quad H^0(\mathcal{O}(i, j)) = S^i V_1^* \otimes S^j V_2^*.$$

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We fix a basis $\{x, y\}$ of V_1^* and a basis $\{z, w\}$ of V_2^* . For a sheaf \mathcal{F} we denote by $[\mathcal{F}]$ its S-equivalence class. If \mathcal{F} is stable, then $[\mathcal{F}]$ is its isomorphism class.

Theorem 1.1. *The variety \mathbf{M} is rational. We have a decomposition of \mathbf{M} into an open subvariety \mathbf{M}_0 , a closed smooth irreducible subvariety \mathbf{M}_1 of codimension 1, and a closed subvariety $\mathbf{M}_2 \cup \mathbf{M}_3$ having two smooth irreducible components $\mathbf{M}_2, \mathbf{M}_3$ of codimension 2, respectively, 3. The subvarieties are defined as follows: $\mathbf{M}_0 \subset \mathbf{M}$ is the subset of sheaves \mathcal{F} having a resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

where φ_{11} and φ_{12} define a subscheme of length 2 of $\mathbb{P}^1 \times \mathbb{P}^1$; $\mathbf{M}_1 \subset \mathbf{M}$ is the subset of sheaves \mathcal{F} having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\varphi_{11} \neq 0, \varphi_{12} \neq 0$; \mathbf{M}_2 is the set of twisted structure sheaves $\mathcal{O}_C(0, 1)$ for a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 3)$; \mathbf{M}_3 is the set of non-split extensions of \mathcal{O}_L by \mathcal{O}_Q for a line $L \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$ and a quartic $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$.

The subvariety \mathbf{M}_1 is isomorphic to a \mathbb{P}^9 -bundle over $\mathbb{P}^1 \times \mathbb{P}^2$ and is the Brill-Noether locus of sheaves \mathcal{F} satisfying $H^0(\mathcal{F}(-1, 1)) \neq 0$ (for $\mathcal{F} \in \mathbf{M}_1$ we have $H^0(\mathcal{F}(-1, 1)) \simeq \mathbb{C}$); \mathbf{M}_2 is isomorphic to \mathbb{P}^{11} and is the Brill-Noether locus of sheaves \mathcal{F} satisfying $H^1(\mathcal{F}) \neq 0$ (for $\mathcal{F} \in \mathbf{M}_2$ we have $H^1(\mathcal{F}) \simeq \mathbb{C}$); \mathbf{M}_3 is isomorphic to a \mathbb{P}^1 -bundle over $\mathbb{P}^8 \times \mathbb{P}^1$.

The proof of this theorem is distributed among the statements of Section 4.

As an application of our classification of sheaves we compute the Betti numbers of \mathbf{M} . For a projective variety X we define the Poincaré polynomial

$$P(X)(\xi) = \sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \xi^{i/2}.$$

The varieties occurring in this paper will have no odd cohomology, so the above will be a genuine polynomial expression.

Theorem 1.2. *The integral homology groups of \mathbf{M} have no torsion. The Poincaré polynomial of \mathbf{M} is*

$$\xi^{13} + 3\xi^{12} + 8\xi^{11} + 10\xi^{10} + 11\xi^9 + 11\xi^8 + 11\xi^7 + 11\xi^6 + 11\xi^5 + 11\xi^4 + 10\xi^3 + 8\xi^2 + 3\xi + 1.$$

The proof of this theorem takes up Section 5 and is based on the approach of Choi and Chung [3], where they study moduli spaces of α -semi-stable pairs and their variation when the parameter α changes. Thus, we show that \mathbf{M} is obtained from the relative Hilbert scheme of two points on the general curve of bidegree $(2, 3)$ by performing one blowing up followed by two blowing down operations. The Betti numbers of \mathbf{M} have already been computed in [4, Section 9.2] in the context of physics. Our calculation agrees with the one in [4]. The Euler characteristic of \mathbf{M} is 110.

In Section 3 we prove that $H^1(\mathcal{F}) = 0$ for $\mathcal{F} \in \mathbf{M} \setminus \mathbf{M}_2$, which is a crucial step in our classification of sheaves. In Section 2 we present our main technical tool: a spectral sequence converging to a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ reminiscent to the Beilinson spectral sequence on the projective plane.

2. PRELIMINARIES

According to [2, Lemma 1], for a given coherent sheaf \mathcal{F} on $\mathbb{P}^1 \times \mathbb{P}^1$ there is a spectral sequence converging to \mathcal{F} in degree zero and to 0 in degrees different from zero. The sheaves E_1^{ij} on the first level E_1 are defined as follows:

$$\begin{aligned} E_1^{ij} &= 0 \quad \text{for } i > 0 \text{ and } i < -2, \\ E_1^{0j} &= H^j(\mathcal{F}) \otimes \mathcal{O}, \\ E_1^{-2,j} &= H^j(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1). \end{aligned}$$

The sheaves $E_1^{-1,j}$ fit into exact sequences

$$H^j(\mathcal{F}(0, -1)) \otimes \mathcal{O}(0, -1) \longrightarrow E_1^{-1,j} \longrightarrow H^j(\mathcal{F}(-1, 0)) \otimes \mathcal{O}(-1, 0).$$

For a sheaf \mathcal{F} with support of dimension 1, which will be our case, the relevant part of E_1 is represented in the tableau

$$(1) \quad H^1(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1) \xrightarrow{\varphi_1} E_1^{-1,1} \xrightarrow{\varphi_2} H^1(\mathcal{F}) \otimes \mathcal{O}$$

$$H^0(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1) \xrightarrow{\varphi_3} E_1^{-1,0} \xrightarrow{\varphi_4} H^0(\mathcal{F}) \otimes \mathcal{O}$$

where the middle sheaves are part of the exact sequences

$$(2) \quad H^0(\mathcal{F}(0, -1)) \otimes \mathcal{O}(0, -1) \longrightarrow E_1^{-1,0} \longrightarrow H^0(\mathcal{F}(-1, 0)) \otimes \mathcal{O}(-1, 0),$$

$$(3) \quad H^1(\mathcal{F}(0, -1)) \otimes \mathcal{O}(0, -1) \longrightarrow E_1^{-1,1} \longrightarrow H^1(\mathcal{F}(-1, 0)) \otimes \mathcal{O}(-1, 0).$$

The relevant part of the second level of the spectral sequence is represented in the tableau

$$\begin{array}{ccc} \text{Ker}(\varphi_1) & \xrightarrow{\quad \text{Ker}(\varphi_2)/\text{Im}(\varphi_1) \quad} & \text{Coker}(\varphi_2) \\ & \searrow \varphi_5 & \\ \text{Ker}(\varphi_3) & \xrightarrow{\quad \text{Ker}(\varphi_4)/\text{Im}(\varphi_3) \quad} & \text{Coker}(\varphi_4) \end{array}$$

The spectral sequence degenerates at $E_3 = E_\infty$. The convergence of the spectral sequence implies that φ_2 is surjective and that we have the exact sequence

$$(4) \quad 0 \longrightarrow \text{Ker}(\varphi_1) \xrightarrow{\varphi_5} \text{Coker}(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow \text{Ker}(\varphi_2)/\text{Im}(\varphi_1) \longrightarrow 0.$$

Let \mathcal{E} be a semi-stable sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with $P_{\mathcal{E}}(m, n) = rm + n + 1$. According to [1, Proposition 11], \mathcal{E} has resolution

$$(5) \quad 0 \longrightarrow \mathcal{O}(-1, -r) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Let \mathcal{E} be a semi-stable sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with $P_{\mathcal{E}}(m, n) = m + sn + 1$. Then \mathcal{E} has resolution

$$(6) \quad 0 \longrightarrow \mathcal{O}(-s, -1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

According to [1, Proposition 14], a semi-stable sheaf \mathcal{E} on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $2m + 2n + 1$ has resolution

$$(7) \quad 0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) \longrightarrow \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

For a sheaf \mathcal{F} of dimension 1, without zero-dimensional torsion, on $\mathbb{P}^1 \times \mathbb{P}^1$ we define the *dual sheaf*

$$\mathcal{F}^\mathcal{D} = \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}}^1(\mathcal{F}, \omega_{\mathbb{P}^1 \times \mathbb{P}^1}).$$

Lemma 2.1. *The map $[\mathcal{F}] \mapsto [\mathcal{F}^\mathcal{D}]$ is well-defined and gives an isomorphism*

$$\mathbf{M}(rm + sn + t) \longrightarrow \mathbf{M}(rm + sn - t).$$

Proof. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Then the dual of \mathcal{F} as a sheaf on \mathbb{P}^3 is compatible with the dual of \mathcal{F} as a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathcal{F}^\mathcal{D} \simeq \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^3}}^2(\mathcal{F}, \omega_{\mathbb{P}^3})|_{\mathbb{P}^1 \times \mathbb{P}^1}.$$

This allows us to apply [15, Theorem 13] to obtain the conclusion. \square

In particular, $\mathbf{M} \simeq \mathbf{M}(3m + 2n - 1)$. Note that the same argument applies for moduli spaces of one-dimensional sheaves on smooth projective varieties.

Theorem 2.2. *We have a decomposition of $\mathbf{M}(3m + 2n - 1)$ into subsets $\mathbf{M}_0^\mathcal{D}$, $\mathbf{M}_1^\mathcal{D}$, $\mathbf{M}_2^\mathcal{D} \cup \mathbf{M}_3^\mathcal{D}$, where $\mathbf{M}_i^\mathcal{D}$ is the image of \mathbf{M}_i under the above isomorphism. Thus, $\mathbf{M}_0^\mathcal{D}$ is the subset of sheaves \mathcal{F} having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-2, -1) \xrightarrow{\varphi} 2\mathcal{O}(-1, 0) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} and φ_{22} define a zero-dimensional subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$; $\mathbf{M}_1^\mathcal{D}$ is the subset of sheaves \mathcal{F} having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -1) \xrightarrow{\varphi} \mathcal{O}(-1, 1) \oplus \mathcal{O}(0, -1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} \neq 0$, $\varphi_{22} \neq 0$; $\mathbf{M}_2^\mathcal{D}$ is the set of structure sheaves of curves of bidegree $(2, 3)$; $\mathbf{M}_3^\mathcal{D}$ is the set of non-split extensions of \mathcal{O}_Q by $\mathcal{O}_L(-2, -1)$ with L a line of bidegree $(0, 1)$ and Q a quartic of bidegree $(2, 2)$.

3. VANISHING OF COHOMOLOGY

The following lemma is analogous to [14, Lemma 6.7]. We will use the word *curve* to denote a subscheme defined by a polynomial equation.

Lemma 3.1. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve and $\mathcal{I} \subset \mathcal{O}_C$ an ideal sheaf. Then there is a curve $C' \subset C$ such that the ideal sheaf of C' in C , denoted \mathcal{I}' , contains \mathcal{I} , and \mathcal{I}'/\mathcal{I} has support of dimension at most 0.*

The following proposition is a strengthening of [1, Lemma 9].

Proposition 3.2. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (s, r) . Then \mathcal{O}_C is semi-stable. If $r > 0$ and $s > 0$, then \mathcal{O}_C is stable.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_C$ be a proper subsheaf and let \mathcal{I}' and C' be as in Lemma 3.1. Let t be the length of \mathcal{I}'/\mathcal{I} and let (s', r') be the bidegree of C' . The Hilbert polynomial of \mathcal{I} is given by

$$\begin{aligned} P_{\mathcal{I}} &= P_{\mathcal{I}'} - t = P_{\mathcal{O}_C} - P_{\mathcal{O}_{C'}} - t \\ &= rm + sn + r + s - rs - r'm - s'n - r' - s' + r's' - t. \end{aligned}$$

Thus, the slopes of \mathcal{I} and \mathcal{O}_C are given by

$$p(\mathcal{I}) = 1 + \frac{r's' - rs - t}{r - r' + s - s'}, \quad p(\mathcal{O}_C) = 1 - \frac{rs}{r + s}.$$

The inequality $p(\mathcal{I}) \leq p(\mathcal{O}_C)$ follows from the inequality $0 \leq rr'(s - s') + ss'(r - r')$. If $r > 0$ and $s > 0$, then this inequality is strict because either $r' < r$ or $s' < s$. \square

Proposition 3.3. *Let \mathcal{F} be a semi-stable sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $P_{\mathcal{F}}(m, n) = rm + sn + t$. Let i and j be integers.*

- (i) *If $\max\{i, j\} < 1 - \frac{rs+t}{r+s}$, then $H^0(\mathcal{F}(i, j)) = 0$.*
- (ii) *If $\min\{i, j\} > -1 + \frac{rs-t}{r+s}$, then $H^1(\mathcal{F}(i, j)) = 0$.*

Proof. Assume that $H^0(\mathcal{F}(i, j)) \neq 0$. Then there is a non-zero morphism $\alpha: \mathcal{O}_D \rightarrow \mathcal{F}(i, j)$ for a curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{J} = \text{Ker}(\alpha)$. By Lemma 3.1 there is a curve $C \subset D$ such that the ideal sheaf \mathcal{I} of C in \mathcal{O}_D contains \mathcal{J} and \mathcal{I}/\mathcal{J} is supported on finitely many points. Since $\mathcal{F}(i, j)$ has no zero-dimensional torsion, $\alpha(\mathcal{I}/\mathcal{J}) = 0$, hence $\mathcal{J} = \mathcal{I}$, and hence α factors through an injective morphism $\mathcal{O}_C \rightarrow \mathcal{F}(i, j)$. From the semi-stability of \mathcal{F} we get the inequality

$$p(\mathcal{O}_C(-i, -j)) = 1 - \frac{r's' + r'i + s'j}{r' + s'} \leq \frac{t}{r + s} = p(\mathcal{F}).$$

Combining this with the inequalities

$$-\frac{rs}{r+s} \leq -\frac{r's'}{r'+s'}, \quad -\max\{i, j\} \leq -\frac{r'i + s'j}{r' + s'}$$

we obtain the inequality

$$1 - \frac{rs}{r+s} - \max\{i, j\} \leq \frac{t}{r+s}.$$

This contradicts the hypothesis of (i). Part (ii) follows from (i) and Serre duality. We have

$$H^1(\mathcal{F}(i, j)) \simeq (H^0(\mathcal{F}^\vee(-i, -j)))^*$$

and, by Lemma 2.1, \mathcal{F}^\vee is semi-stable with Hilbert polynomial $rm + sn - t$. Thus, the right-hand-side vanishes if

$$\max\{-i, -j\} < 1 - \frac{rs-t}{r+s}. \quad \square$$

Using this proposition we can give another proof to the fact shown at [1, Proposition 10] that there are no semi-stable sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $rm + t$, for $r \geq 2$ and t not a multiple of r .

Corollary 3.4. *The moduli spaces $M(rm + t)$ are empty for $r \geq 2$ and $0 < t < r$.*

Proof. Assume that \mathcal{F} is a semi-stable sheaf in one of these moduli spaces. From Proposition 3.3 (i) we get $H^0(\mathcal{F}) = 0$. From Proposition 3.3 (ii) we get $H^1(\mathcal{F}) = 0$. Thus, $t = \chi(\mathcal{F}) = 0$, which contradicts our choice of t . \square

Proposition 3.5. *For $\mathcal{F} \in \mathbf{M}$ we have $H^0(\mathcal{F}(-1, -1)) = 0$, $H^0(\mathcal{F}(-1, 0)) = 0$, and $H^0(\mathcal{F}(0, -1)) \neq 0$ if and only if $\mathcal{F} \simeq \mathcal{O}_C(0, 1)$ for a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 3)$.*

Proof. The vanishing of $H^1(\mathcal{F}(-1, -1))$ follows from Proposition 3.3 (i). Assume that $H^0(\mathcal{F}(i, j)) \neq 0$, where $(i, j) = (-1, 0)$ or $(0, -1)$. As in the proof of Proposition 3.3, there is a curve C and an injective morphism $\mathcal{O}_C \rightarrow \mathcal{F}(i, j)$. In Table 1 below we have the possible bidegrees of C and the slopes of $\mathcal{O}_C(-i, -j)$.

The only case in which $\mathcal{O}_C(-i, -j)$ does not violate the semi-stability of \mathcal{F} is when $\deg(C) = (2, 3)$ and $(i, j) = (0, -1)$. We deduce that $H^0(\mathcal{F}(-1, 0)) = 0$ and, if $H^0(\mathcal{F}(0, -1)) \neq 0$, then $\mathcal{F} \simeq \mathcal{O}_C(0, 1)$ for a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 3)$.

Table 1. Possibilities for C .

$\deg(C)$	$P_{\mathcal{O}_C}$	$p(\mathcal{O}_C(1, 0))$	$p(\mathcal{O}_C(0, 1))$
(2, 3)	$3m + 2n - 1$	$2/5$	$1/5$
(2, 2)	$2m + 2n$	$1/2$	$1/2$
(1, 3)	$3m + n + 1$	1	$1/2$
(2, 1)	$m + 2n + 1$	$2/3$	1
(1, 2)	$2m + n + 1$	1	$2/3$
(0, 3)	$3m + 3$	2	1
(2, 0)	$2n + 2$	1	2
(1, 1)	$m + n + 1$	1	1
(0, 2)	$2m + 2$	2	1
(1, 0)	$n + 1$	1	2
(0, 1)	$m + 1$	2	1

It remains to show that $\mathcal{O}_C(0, 1)$ is semi-stable. Let $\mathcal{I} \subset \mathcal{O}_C$ be an ideal sheaf and let \mathcal{I}' and C' be as in Lemma 3.1. In Table 2 below we have the possible bidegrees of C' and the resulting slopes of $\mathcal{I}'(0, 1)$.

Table 2. Possibilities for C' .

$\deg(C')$	$P_{\mathcal{O}_{C'}}$	$P_{\mathcal{I}'}$	$p(\mathcal{I}'(0, 1))$
(2, 2)	$2m + 2n$	$m - 1$	-1
(1, 3)	$3m + n + 1$	$n - 2$	-1
(2, 1)	$m + 2n + 1$	$2m - 2$	-1
(1, 2)	$2m + n + 1$	$m + n - 2$	$-1/2$
(0, 3)	$3m + 3$	$2n - 4$	-1
(2, 0)	$2n + 2$	$3m - 3$	-1
(1, 1)	$m + n + 1$	$2m + n - 2$	$-1/3$
(0, 2)	$2m + 2$	$m + 2n - 3$	$-1/3$
(1, 0)	$n + 1$	$3m + n - 2$	$-1/4$
(0, 1)	$m + 1$	$2m + 2n - 2$	0

In all cases $p(\mathcal{I}'(0, 1)) < p(\mathcal{O}_C(0, 1))$. In conclusion, $\mathcal{O}_C(0, 1)$ is semi-stable. \square

Proposition 3.6. *Let \mathcal{F} give a point in \mathbf{M} . If $H^0(\mathcal{F}(0, -1)) = 0$, then $H^1(\mathcal{F}) = 0$.*

Proof. Denote $d = \dim H^1(\mathcal{F})$. In view of Proposition 3.5 and exact sequence (2) we get $E_1^{-1,0} = 0$. Thus, the exact sequence (4) becomes

$$0 \longrightarrow \mathcal{Ker}(\varphi_1) \xrightarrow{\varphi_5} (d+1)\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Ker}(\varphi_2)/\mathcal{Im}(\varphi_1) \longrightarrow 0.$$

From Proposition 3.5 we get $H^1(\mathcal{F}(-1, -1)) \simeq \mathbb{C}^4$, hence we have the exact sequence

$$0 \longrightarrow \mathcal{Ker}(\varphi_1) \longrightarrow 4\mathcal{O}(-1, -1) \longrightarrow \mathcal{Im}(\varphi_1) \longrightarrow 0.$$

We also have the exact sequence

$$0 \longrightarrow \mathcal{Ker}(\varphi_2) \longrightarrow E_1^{-1,1} \longrightarrow d\mathcal{O} \longrightarrow 0.$$

From these exact sequences we can compute the Hilbert polynomial of $E_1^{-1,1}$:

$$\begin{aligned} P_{E_1^{-1,1}} &= P_{\mathcal{K}er(\varphi_2)} + dP_{\mathcal{O}} = P_{\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)} + P_{\mathcal{I}m(\varphi_1)} + dP_{\mathcal{O}} \\ &= P_{\mathcal{F}} - (d+1)P_{\mathcal{O}} + P_{\mathcal{K}er(\varphi_1)} + 4P_{\mathcal{O}(-1,-1)} - P_{\mathcal{K}er(\varphi_1)} + dP_{\mathcal{O}} \\ &= P_{\mathcal{F}} - P_{\mathcal{O}} + 4P_{\mathcal{O}(-1,-1)} \\ &= 3mn + 2m + n. \end{aligned}$$

The exact sequence (3) becomes

$$\mathcal{O}(0, -1) \longrightarrow E_1^{-1,1} \longrightarrow 2\mathcal{O}(-1, 0).$$

Since $P_{E_1^{-1,1}} = P_{\mathcal{O}(0,-1)} + 2P_{\mathcal{O}(-1,0)}$ this sequence is also exact on the left and right, and, in fact, it is split exact. We deduce that $E_1^{-1,1} \simeq \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0)$. It follows that $d \leq 2$ because there is, obviously, no surjective morphism

$$\varphi_2: \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0) \longrightarrow d\mathcal{O}$$

for $d \geq 3$. Assume that $d = 2$. Then the maximal minors of φ_2 have no common factor, otherwise φ_2 would not be surjective. It follows that $\mathcal{K}er(\varphi_2) \simeq \mathcal{O}(-2, -1)$. We have

$$P_{\mathcal{C}oker(\varphi_2)} = 2P_{\mathcal{O}} - P_{\mathcal{O}(0,-1)} - 2P_{\mathcal{O}(-1,0)} + P_{\mathcal{O}(-2,-1)} = 2$$

which contradicts the surjectivity of φ_2 . Assume that $d = 1$. If the restriction of φ_2 to $\mathcal{O}(0, -1)$ were zero, then $\mathcal{K}er(\varphi_2) \simeq \mathcal{O}(0, -1) \oplus \mathcal{O}(-2, 0)$. This would yield a contradiction because $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$ would contain $\mathcal{O}(-2, 0)$ as a direct summand, but there is no surjective morphism $\mathcal{F} \rightarrow \mathcal{O}(-2, 0)$. Thus, we may write

$$\begin{aligned} \varphi_2 &= \begin{bmatrix} -1 \otimes z & x \otimes 1 & y \otimes 1 \end{bmatrix}, \\ \varphi_1 &= \begin{bmatrix} x \otimes 1 & y \otimes 1 & 0 & 0 \\ 1 \otimes z & 0 & 0 & 0 \\ 0 & 1 \otimes z & 0 & 0 \end{bmatrix}. \end{aligned}$$

It follows that $\mathcal{K}er(\varphi_1) \simeq 2\mathcal{O}(-1, -1)$. Thus, $\mathcal{C}oker(\varphi_1)$ has Hilbert polynomial $2P_{\mathcal{O}} - 2P_{\mathcal{O}(-1,-1)} = 2m + 2n + 2$, hence it has slope $1/2$, and hence it is a destabilizing subsheaf of \mathcal{F} . In conclusion, $d = 0$. \square

4. CLASSIFICATION OF SHEAVES

Assume that \mathcal{F} gives a point in \mathbf{M} and that $H^0(\mathcal{F}(0, -1)) = 0$. Then, as seen at Proposition 3.6, $H^1(\mathcal{F}) = 0$, and, as seen in the proof of this proposition, $E_1^{-1,1} \simeq \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0)$. Thus, the exact sequence (4) becomes

$$(8) \quad 0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0,$$

where

$$\varphi_1: 4\mathcal{O}(-1, -1) \longrightarrow \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0).$$

Lemma 4.1. *Assume that \mathcal{F} gives a point in \mathbf{M} and that $H^0(\mathcal{F}(0, -1)) = 0$. Assume that the maximal minors of φ_1 have no common factor. Then $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-2, -3)$ and $\mathcal{C}oker(\varphi_1)$ is isomorphic to the structure sheaf of a zero-dimensional subscheme $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ of length 2. Moreover, Z is not contained in a line of bidegree $(0, 1)$. Thus, we have a non-split extension*

$$(9) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where C is a curve of bidegree $(2, 3)$ containing Z .

Proof. Let ζ_j , $1 \leq j \leq 4$, be the maximal minor of φ_1 obtained by deleting column j , for a matrix representation of φ_1 . It is well-known that the sequence

$$0 \longrightarrow \mathcal{O}(-2, -3) \xrightarrow{\zeta} 4\mathcal{O}(-1, -1) \xrightarrow{\varphi_1} \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0),$$

$$\zeta = \begin{bmatrix} \zeta_1 & -\zeta_2 & \zeta_3 & -\zeta_4 \end{bmatrix}^T,$$

is exact. Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the subscheme given by the ideal $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. The Hilbert polynomial of \mathcal{O}_Z can be computed from the exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus 2\mathcal{O}(-1, -3) \xrightarrow{\varphi_1^T} 4\mathcal{O}(-1, -2) \xrightarrow{\zeta^T} \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We get $P_{\mathcal{O}_Z} = 2$, hence Z is zero-dimensional of length 2. From the short exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{H}om(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{H}om(\mathcal{O}, \mathcal{O}) \longrightarrow \mathcal{H}om(\mathcal{I}_Z, \mathcal{O}) \\ &\longrightarrow \mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{E}xt^1(\mathcal{O}, \mathcal{O}) \longrightarrow \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}) \\ &\longrightarrow \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{E}xt^2(\mathcal{O}, \mathcal{O}). \end{aligned}$$

The sheaves $\mathcal{H}om(\mathcal{O}_Z, \mathcal{O})$, $\mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O})$, $\mathcal{E}xt^1(\mathcal{O}, \mathcal{O})$, $\mathcal{E}xt^2(\mathcal{O}, \mathcal{O})$ are zero, hence we get the isomorphisms

$$\mathcal{H}om(\mathcal{I}_Z, \mathcal{O}) \simeq \mathcal{O}, \quad \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}) \simeq \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \simeq \mathcal{O}_Z.$$

We apply the long $\mathcal{E}xt(-, \mathcal{O})$ -sequence to the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus 2\mathcal{O}(-1, -3) \xrightarrow{\varphi_1^T} 4\mathcal{O}(-1, -2) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

and we use the above isomorphisms to obtain the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow 4\mathcal{O}(1, 2) \xrightarrow{\psi} \mathcal{O}(2, 2) \oplus 2\mathcal{O}(1, 3) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

The morphism ψ is a twist of φ_1 . Assume that Z were contained in a line of bidegree $(0, 1)$. Then we would have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-2, -2) \oplus 2\mathcal{O}(-1, -3) & \longrightarrow & 4\mathcal{O}(-1, -2) & \longrightarrow & \mathcal{I}_Z \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \parallel \\ 0 & \longrightarrow & \mathcal{O}(-2, -1) & \longrightarrow & \mathcal{O}(-2, 0) \oplus \mathcal{O}(0, -1) & \longrightarrow & \mathcal{I}_Z \longrightarrow 0 \end{array}$$

in which $\alpha \neq 0$. Thus $\text{rank}(\mathcal{K}er(\alpha)) = 3$, hence $\beta = 0$, and hence $\mathcal{C}oker(\beta) \simeq \mathcal{O}(-2, -1)$ contains $\mathcal{O}(-2, 0)$ as a direct summand. This is absurd.

The exact sequence (9) follows from (8) with $\mathcal{O}_C = \mathcal{C}oker(\varphi_5)$. From sequence (9), and since \mathcal{F} has no zero-dimensional torsion, we see that \mathcal{F} has schematic support C , hence Z is contained in C . \square

Lemma 4.2. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree $(2, 3)$ and let $Z \subset C$ be a zero-dimensional subscheme of length 2. Let \mathcal{F} be an extension of \mathcal{O}_Z by \mathcal{O}_C that has no zero-dimensional torsion. Then \mathcal{F} is uniquely determined up to isomorphism. This means that if \mathcal{G} is another extension of \mathcal{O}_Z by \mathcal{O}_C that has no zero-dimensional torsion, then $\mathcal{F} \simeq \mathcal{G}$.*

Proof. By Serre duality $\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C) \simeq (\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_Z))^*$. From the short exact sequence

$$(10) \quad 0 \longrightarrow \mathcal{O}(-2, -3) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}(\mathcal{O}_C, \mathcal{O}_Z) \simeq H^0(\mathcal{O}_Z) \simeq \mathbb{C}^2 &\longrightarrow \text{Hom}(\mathcal{O}, \mathcal{O}_Z) \simeq \mathbb{C}^2 \\ &\longrightarrow \text{Hom}(\mathcal{O}(-2, -3), \mathcal{O}_Z) \simeq \mathbb{C}^2 \longrightarrow \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_Z) \longrightarrow \text{Ext}^1(\mathcal{O}, \mathcal{O}_Z) = 0 \end{aligned}$$

We obtain $\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C) \simeq \mathbb{C}^2$.

Assume that $Z = \{p, q\}$ for distinct points $p, q \in C$. We denote by \mathbb{C}_p and \mathbb{C}_q the structure sheaves of the subschemes $\{p\}$, respectively, $\{q\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. From sequence (10) we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}(\mathcal{O}_C, \mathbb{C}_p) \simeq \mathbb{C} &\longrightarrow \text{Hom}(\mathcal{O}, \mathbb{C}_p) \simeq \mathbb{C} \longrightarrow \text{Hom}(\mathcal{O}(-2, -3), \mathbb{C}_p) \simeq \mathbb{C} \\ &\longrightarrow \text{Ext}^1(\mathcal{O}_C, \mathbb{C}_p) \simeq (\text{Ext}^1(\mathbb{C}_p, \mathcal{O}_C))^* \longrightarrow \text{Ext}^1(\mathcal{O}, \mathbb{C}_p) = 0. \end{aligned}$$

Thus, there is a unique non-trivial extension of \mathbb{C}_p by \mathcal{O}_C , denoted by \mathcal{E} . From the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

we get the long exact sequence

$$0 = \text{Hom}(\mathbb{C}_q, \mathbb{C}_p) \longrightarrow \text{Ext}^1(\mathbb{C}_q, \mathcal{O}_C) \simeq \mathbb{C} \longrightarrow \text{Ext}^1(\mathbb{C}_q, \mathcal{E}) \longrightarrow \text{Ext}^1(\mathbb{C}_q, \mathbb{C}_p) = 0.$$

Thus, there is a unique non-trivial extension of \mathbb{C}_q by \mathcal{E} , hence \mathcal{F} is unique up to isomorphism.

We next consider the case when Z is a double point supported on $p \in C$. We construct a resolution of \mathcal{E} by combining resolution (10) with the resolution

$$0 \longrightarrow \mathcal{O}(-2, -3) \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \longrightarrow \mathcal{O}(-1, -2) \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

The map $\mathcal{O}(-1, -2) \rightarrow \mathbb{C}_p$ lifts to \mathcal{E} because $H^1(\mathcal{O}_C(1, 2)) = 0$. Applying the argument at the proof of [16, Proposition 2.3.2], which uses the fact that $\text{Ext}^1(\mathbb{C}_p, \mathcal{O}) = 0$, we can show that the induced map $\mathcal{O}(-2, -3) \rightarrow \mathcal{O}(-2, -3)$ is non-zero. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \xrightarrow{\psi} \mathcal{O}(-1, -2) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

in which $\psi_{11} \neq 0$, $\psi_{12} \neq 0$, $\psi_{11}(p) = 0$, $\psi_{12}(p) = 0$. Moreover, $\psi_{21}(p) = 0$ and $\psi_{22}(p) = 0$ if and only if p is a singular point of C . We have the exact sequence

$$\begin{aligned} \text{Hom}(\mathcal{O}(-1, -2) \oplus \mathcal{O}, \mathbb{C}_p) \simeq \mathbb{C}^2 &\xrightarrow{\psi(p)} \text{Hom}(\mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3), \mathbb{C}_p) \simeq \mathbb{C}^2 \\ &\longrightarrow \text{Ext}^1(\mathcal{E}, \mathbb{C}_p) \simeq (\text{Ext}^1(\mathbb{C}_p, \mathcal{E}))^* \longrightarrow \text{Ext}^1(\mathcal{O}(-1, -2) \oplus \mathcal{O}, \mathbb{C}_p) = 0. \end{aligned}$$

We get a unique non-trivial extension of \mathbb{C}_p by \mathcal{E} if p is a regular point of C . In this case \mathcal{F} is unique up to isomorphism.

Assume now that p is a singular point of C . Then $\psi(p) = 0$, hence $\text{Ext}^1(\mathbb{C}_p, \mathcal{E}) \simeq \mathbb{C}^2$. According to [11, Proposition 2.3.1], the subset $U_Z \subset \mathbb{P}(\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C)) \simeq \mathbb{P}^1$ of extension sheaves having no zero-dimensional torsion is open. We construct a map $v_Z: U_Z \rightarrow \mathbb{P}(\text{Ext}^1(\mathbb{C}_p, \mathcal{E})) \simeq \mathbb{P}^1$ as follows. Let \mathcal{I} be the ideal sheaf of $\{p\}$ in

Z . Note that $\mathcal{I} \simeq \mathbb{C}_p$ as modules over \mathcal{O} . Given $\mathcal{F} \in U_Z$ let \mathcal{A} be the pull-back in \mathcal{F} of \mathcal{I} . Then there is a unique isomorphism $\mathcal{E} \rightarrow \mathcal{A}$ making the diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{A} & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \end{array}$$

The composite map $\mathcal{E} \rightarrow \mathcal{A} \rightarrow \mathcal{F}$ has cokernel \mathbb{C}_p , so \mathcal{F} is an extension of \mathbb{C}_p by \mathcal{E} . We claim that the image of v_Z is a point. If we can prove this claim, then it will follow that \mathcal{F} is uniquely determined up to isomorphism. Assume that the image of v_Z is an open subset of \mathbb{P}^1 . The zero-dimensional schemes Z' of length 2 supported on p are parametrized by \mathbb{P}^1 . Thus there is $Z' \neq Z$ such that $v_{Z'}(U_{Z'}) \cap v_Z(U_Z) \neq \emptyset$. This means that we have extensions $\mathcal{F} \in U_Z$, $\mathcal{F}' \in U_{Z'}$, and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \\ & & \parallel & & \downarrow \simeq & & \parallel \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \end{array}$$

The isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ fits into a commutative square

$$\begin{array}{ccc} \mathcal{O}_C & \longrightarrow & \mathcal{F} \\ \parallel & & \downarrow \simeq \\ \mathcal{O}_C & \longrightarrow & \mathcal{F}' \end{array}$$

We get an induced isomorphism of cokernels $\mathcal{O}_Z \rightarrow \mathcal{O}_{Z'}$, which contradicts our choice of Z' . In conclusion, the image of v_Z is a point. \square

The difficult case in the previous lemma is when Z is concentrated in one point. For this case we will give an alternate more general argument. The following lemma and its proof were provided by Jean-Marc Dr  zet, to whom the author is grateful.

Lemma 4.3. *Let S be a smooth projective surface and $C \subset S$ a Cohen-Macaulay curve. Let $Z \subset C$ be a zero-dimensional subscheme of length 2 concentrated on a single point p , and \mathcal{L} a line bundle on C . Then there exists an extension*

$$(11) \quad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where \mathcal{F} has no zero-dimensional torsion. The sheaf \mathcal{F} is unique up to isomorphism.

Proof. The extensions (11) on C and on S are the same. Indeed, by [7, Proposition 2.2.1] we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_Z, \mathcal{L}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_S}^1(\mathcal{O}_Z, \mathcal{L}) \longrightarrow \mathrm{Hom}(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_C), \mathcal{L}).$$

The group on the right vanishes because $\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_C)$ is supported on Z , yet \mathcal{L} has no zero-dimensional torsion. From Serre duality we have

$$(12) \quad \mathrm{Ext}_{\mathcal{O}_S}^1(\mathcal{O}_Z, \mathcal{L}) \simeq \mathrm{Ext}_{\mathcal{O}_S}^1(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S)^*.$$

Again from [7, Proposition 2.2.1] we have the exact sequence

$$\begin{aligned} \mathrm{Ext}_{\mathcal{O}_C}^1(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) &\longrightarrow \mathrm{Ext}_{\mathcal{O}_S}^1(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) \longrightarrow \mathrm{Hom}(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_C), \mathcal{O}_Z \otimes \omega_S) \\ &\longrightarrow \mathrm{Ext}_{\mathcal{O}_C}^2(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S). \end{aligned}$$

The first and the last groups vanish, hence we obtain the functorial isomorphisms

$$\begin{aligned} \mathrm{Ext}_{\mathcal{O}_S}^1(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) &\simeq \mathrm{Hom}(\mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_C), \mathcal{O}_Z \otimes \omega_S) \\ &\simeq \mathrm{Hom}(\mathcal{L}(-C), \mathcal{O}_Z \otimes \omega_S) \\ &\simeq H^0((\mathcal{L}^*(C) \otimes \omega_S)|_Z) \simeq \mathbb{C}^2. \end{aligned}$$

Now consider an extension (11) which is non-split, and suppose that \mathcal{F} has a zero-dimensional subsheaf \mathcal{T} . Since \mathcal{L} is torsion-free on C the composition $\mathcal{T} \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Z$ is injective. There are only two non-zero subsheaves of \mathcal{O}_Z : the sheaf of sections vanishing at p , which is isomorphic to \mathbb{C}_p , and \mathcal{O}_Z itself. Since the extension is non-split, we have $\mathcal{T} = \mathbb{C}_p$. Let $\mathcal{G} = \mathcal{F}/\mathcal{T}$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{C}_p & \xlongequal{\quad} & \mathbb{C}_p & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_Z \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Let $\sigma \in \mathrm{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_Z, \mathcal{L})$ correspond to extension (11) and let $\tau \in \mathrm{Ext}_{\mathcal{O}_C}^1(\mathbb{C}_p, \mathcal{L})$ correspond to the extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

Consider the morphism

$$\Phi: \mathrm{Ext}_{\mathcal{O}_C}^1(\mathbb{C}_p, \mathcal{L}) \longrightarrow \mathrm{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_Z, \mathcal{L})$$

induced by the surjective morphism $\mathcal{O}_Z \rightarrow \mathbb{C}_p$. It is then easy to see that $\Phi(\tau) = \sigma$ (see [6, Proposition 4.3.1]). It follows that for an extension (11) associated to σ , the sheaf \mathcal{F} has zero-dimensional torsion if and only if $\sigma \in \mathrm{Im}(\Phi)$.

According to (12) and to the above functorial isomorphisms, Φ is the transpose of the canonical surjective morphism

$$\begin{array}{ccc} \Psi: H^0((\mathcal{L}^*(C) \otimes \omega_S)|_Z) & \longrightarrow & H^0((\mathcal{L}^*(C) \otimes \omega_S)|_p) \\ \parallel & & \parallel \\ \mathbb{C}^2 & & \mathbb{C} \end{array}$$

The kernel of Ψ is the set $m_p \simeq \mathbb{C}$ of sections vanishing at p . Then $\sigma \in \mathrm{Im}(\Phi)$ if and only if σ vanishes on m_p . The set of extensions σ that do not vanish on m_p is non-empty. This proves the existence part of the lemma. It is easy to check that the group of automorphisms of \mathcal{O}_Z acts transitively on the set of extensions σ that do not vanish on m_p . This proves the uniqueness part of the lemma. \square

Proposition 4.4. *Let \mathcal{F} be an extension as in (9), without zero-dimensional torsion, for a curve C of bidegree $(2, 3)$ and a subscheme $Z \subset C$ that is the intersection of two curves of bidegree $(1, 1)$. Then \mathcal{F} gives a point in \mathbf{M} . Let $\mathbf{M}_0 \subset \mathbf{M}$ be the subset of such sheaves \mathcal{F} . Then \mathbf{M}_0 is open and it can be described as the set of sheaves \mathcal{G} having a resolution of the form*

$$(13) \quad 0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

where φ_{11} and φ_{12} define a zero-dimensional subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Any $\text{Coker}(\varphi)$ is an extension of \mathcal{O}_Z by \mathcal{O}_C without zero dimensional torsion, where $Z = \{\varphi_{11} = 0, \varphi_{12} = 0\}$ and $C = \{\det \varphi = 0\}$, hence it is the unique extension of \mathcal{O}_Z by \mathcal{O}_C that has no zero-dimensional torsion. It remains to show that any sheaf \mathcal{G} having resolution (13) is semi-stable. Assume that \mathcal{G} had a destabilizing subsheaf \mathcal{E} . Without loss of generality we may take \mathcal{E} to be semi-stable. Since $\dim H^0(\mathcal{G}) = 1$, we have $\chi(\mathcal{E}) = 1$. According to Corollary 3.4, \mathcal{E} cannot have Hilbert polynomial $2m + 1$, $2n + 1$, or $3m + 1$. If $P_{\mathcal{E}} = n + 1$, then resolution (5) with $r = 0$ fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1, 0) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & 2\mathcal{O}(-1, -2) & \xrightarrow{\varphi} & \mathcal{O}(0, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

with $\alpha \neq 0$. Since $\beta = 0$ we get $\alpha\psi = 0$, hence $\psi = 0$, which yields a contradiction. We obtain a contradiction in the same manner if $P_{\mathcal{E}} = m + 1, m + n + 1, m + 2n + 1$. Assume that $P_{\mathcal{E}} = 2m + n + 1$. Then resolution (5) with $r = 2$ is part of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1, -2) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & 2\mathcal{O}(-1, -2) & \xrightarrow{\varphi} & \mathcal{O}(0, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Since $\alpha_{11} = 0$ we obtain $\varphi_{11}\beta_{11} + \varphi_{12}\beta_{21} = 0$. This contradicts the fact that φ_{11} and φ_{12} are linearly independent. Assume that $P_{\mathcal{E}} = 3m + n + 1$. Then resolution (5) with $r = 3$ is the first line of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1, -3) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & 2\mathcal{O}(-1, -2) & \xrightarrow{\varphi} & \mathcal{O}(0, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Write $\beta_{11} = 1 \otimes l_1$, $\beta_{21} = 1 \otimes l_2$, $\varphi_{11} = x \otimes u_1 + y \otimes v_1$, $\varphi_{12} = x \otimes u_2 + y \otimes v_2$. From $\alpha_{11} = 0$ we obtain

$$\begin{aligned} 0 &= \varphi_{11}(1 \otimes l_1) + \varphi_{12}(1 \otimes l_2), \\ 0 &= x \otimes (l_1 u_1 + l_2 u_2) + y \otimes (l_1 v_1 + l_2 v_2), \\ 0 &= l_1 u_1 + l_2 u_2, \quad 0 = l_1 v_1 + l_2 v_2, \\ u_1 &= a l_2, \quad u_2 = -a l_1, \quad v_1 = b l_2, \quad v_2 = -b l_1, \\ \varphi_{11} &= (ax + by) \otimes l_2, \quad \varphi_{12} = -(ax + by) \otimes l_1 \end{aligned}$$

for some $a, b \in \mathbb{C}$. This contradicts our hypothesis that φ_{11} and φ_{12} define a zero-dimensional subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Assume, finally, that $P_{\mathcal{E}} = 2m + 2n + 1$. Then resolution (7) fits into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) & \xrightarrow{\psi} & \mathcal{O}(-1, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & 2\mathcal{O}(-1, -2) & \xrightarrow{\varphi} & \mathcal{O}(0, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} \longrightarrow 0 \end{array}$$

We have $\alpha_{22} \neq 0$ because the map $\mathcal{E} \rightarrow \mathcal{G}$ is injective on global sections. It follows that α is injective, otherwise $\text{Ker}(\alpha) \simeq \mathcal{O}(-1, -1)$, but this cannot be a subsheaf of $\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2)$. It follows that β is injective, which is absurd. \square

Corollary 4.5. *The variety \mathbf{M} is rational.*

Proof. Consider the open subset $B \subset \mathbf{M}_0$ given by the condition that Z consist of two distinct points. Notice that B is a bundle with fiber \mathbb{P}^9 and base an open subset of $((\mathbb{P}^1 \times \mathbb{P}^1)^2 \setminus \Delta)/S_2$. Here Δ is the diagonal of the product of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ and S_2 is the group of permutations of two elements. \square

Proposition 4.6. *Let \mathcal{F} be an extension as in (9), that has no zero-dimensional torsion, for a curve C of bidegree $(2, 3)$ and a subscheme $Z \subset C$ that is the intersection of two curves of bidegree $(0, 2)$, respectively, $(1, 0)$. Then \mathcal{F} gives a point in \mathbf{M} . Let $\mathbf{M}_1 \subset \mathbf{M}$ be the subset of such sheaves \mathcal{F} . Then \mathbf{M}_1 is irreducible of codimension 1 and it can be described as the set of sheaves \mathcal{G} having a resolution of the form*

$$(14) \quad 0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\varphi_{11} \neq 0$ and $\varphi_{12} \neq 0$.

Proof. We will show that any sheaf \mathcal{G} having resolution (14) has no destabilizing subsheaves. Assume that \mathcal{G} had a destabilizing subsheaf \mathcal{E} . Without loss of generality we may take \mathcal{E} to be semi-stable. Since $\dim H^0(\mathcal{G}) = 1$, we have $\chi(\mathcal{E}) = 1$. According to Corollary 3.4, \mathcal{E} cannot have Hilbert polynomial $2m + 1$, $2n + 1$, or $3m + 1$. If $P_{\mathcal{E}} = n + 1$, then resolution (5) with $r = 0$ fits into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-1, 0) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) & \xrightarrow{\varphi} & \mathcal{O}(-1, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} \longrightarrow 0 \end{array}$$

with $\alpha \neq 0$. Since $\beta = 0$ we get $\alpha\psi = 0$, hence $\psi = 0$, which yields a contradiction. We obtain a contradiction in the same manner if $P_{\mathcal{E}} = m + 1$, $m + n + 1$, $2m + n + 1$. Assume that $P_{\mathcal{E}} = m + 2n + 1$. Then resolution (6) with $s = 2$ is part of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-2, -1) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) & \xrightarrow{\varphi} & \mathcal{O}(-1, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} \longrightarrow 0 \end{array}$$

Since α is injective on global sections, α is injective, hence β is injective, too, and hence we may write

$$\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{Then} \quad \varphi\beta = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \end{bmatrix} = \alpha\psi = \begin{bmatrix} 0 \\ \psi \end{bmatrix},$$

hence $\varphi_{11} = 0$, which contradicts our hypothesis. We obtain a contradiction in the same manner if $P_{\mathcal{E}} = 3m + n + 1$. Assume, finally, that $P_{\mathcal{E}} = 2m + 2n + 1$. Then resolution (7) is the first line of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) & \xrightarrow{\psi} & \mathcal{O}(-1, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) & \xrightarrow{\varphi} & \mathcal{O}(-1, -1) \oplus \mathcal{O} & \longrightarrow & \mathcal{G} \longrightarrow 0 \end{array}$$

Notice that α and $\alpha(1, 1)$ are injective on global sections, hence α is injective, and hence β is injective, which is absurd. \square

Let W_1 be the set of morphisms φ occurring in resolution (14) and consider the algebraic group

$$G_1 = (\text{Aut}(\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3)) \times \text{Aut}(\mathcal{O}(-1, -1) \oplus \mathcal{O})) / \mathbb{C}^*$$

acting on W_1 by conjugation.

Proposition 4.7. *The variety \mathbf{M}_1 is isomorphic to the geometric quotient W_1/G_1 . Thus, \mathbf{M}_1 is a \mathbb{P}^9 -bundle over $\mathbb{P}^1 \times \mathbb{P}^2$, so it is smooth and closed in \mathbf{M} .*

Proof. The canonical map $W_1 \rightarrow \mathbf{M}_1$, $\varphi \mapsto [\text{Coker}(\varphi)]$, has local sections, and its fibers are the G_1 -orbits, hence it is a geometric quotient map. We construct the local sections as follows. Given $[\mathcal{F}] \in \mathbf{M}_1$, let C be the schematic support of \mathcal{F} , and let Z be the zero-dimensional scheme of length 2 given by the exact sequence (9). Then $Z = L \cap (L_1 \cup L_2)$, where L is a line of bidegree $(1, 0)$, and L_1, L_2 are lines, each of bidegree $(0, 1)$. Choose equations $\varphi_{11} = 0$ of L , $\varphi_{12} = 0$ of $L_1 \cup L_2$, and $f = 0$ of C . Then we can write $f = \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}$ for some $\varphi_{21} \in S^2 V_1^* \otimes V_2^*$, $\varphi_{22} \in V_1^* \otimes S^3 V_2^*$. Map $[\mathcal{F}]$ to the morphism represented by the matrix $(\varphi_{ij})_{1 \leq i, j \leq 2}$. This construction can be done for a local flat family in a neighborhood of $[\mathcal{F}]$ in \mathbf{M}_1 .

We now describe W_1/G_1 . Let $U \subset V_1^* \oplus S^2 V_2^*$ be the open subset

$$\{(\varphi_{11}, \varphi_{12}), \varphi_{11} \neq 0, \varphi_{12} \neq 0\}.$$

Let F be the trivial vector bundle on U with fiber $(S^2 V_1^* \otimes V_2^*) \oplus (V_1^* \otimes S^3 V_2^*)$. Consider the subbundle $E \subset F$ which over the point $(\varphi_{11}, \varphi_{12})$ has fiber $(\varphi_{11} V_1^* \otimes V_2^*) \oplus (\varphi_{12} V_1^* \otimes V_2^*)$. The quotient bundle $G = F/E$ has rank 10 and is linearized for the canonical action of $\mathbb{C}^* \times \mathbb{C}^* = \text{Aut}(\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3))$ on U . Thus, G descends to a vector bundle H over $U/\mathbb{C}^* \times \mathbb{C}^* = \mathbb{P}(V_1^*) \times \mathbb{P}(S^2 V_2^*) \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Clearly, $\mathbb{P}(H) \simeq W_1/G_1$. \square

Proposition 4.8. *Assume that \mathcal{F} gives a point in \mathbf{M} and that $H^0(\mathcal{F}(0, -1)) = 0$. Assume that the maximal minors of φ_1 have a common factor. Then $\text{Ker}(\varphi_1) \simeq \mathcal{O}(-2, -2)$ and $\text{Coker}(\varphi_1) \simeq \mathcal{O}_L$ for a line $L \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. Thus, we have an extension*

$$(15) \quad 0 \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

for a quartic curve $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$. Conversely, any non-split extension of this form is semi-stable. We have $\text{Ext}_{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}}^1(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$.

Proof. Let $g = \gcd(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, where $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are defined as in the proof of Lemma 4.1. We have the exact sequence

$$0 \longrightarrow \mathcal{O}(i, j) \xrightarrow{\eta} 4\mathcal{O}(-1, -1) \xrightarrow{\varphi_1} \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0),$$

$$\eta = \begin{bmatrix} \frac{\zeta_1}{g} & -\frac{\zeta_2}{g} & \frac{\zeta_3}{g} & -\frac{\zeta_4}{g} \end{bmatrix}^T.$$

The possibilities for the kernel of φ_1 are given in Table 3 below.

Table 3. Kernel of φ_1 .

$\deg(g)$	(i, j)	$P_{\text{Coker}(\varphi_5)}$
$(1, 0)$	$(-1, -3)$	$3m + n + 1$
$(0, 1)$	$(-2, -2)$	$2m + 2n$
$(0, 2)$	$(-2, -1)$	$m + 2n + 1$
$(1, 1)$	$(-1, -2)$	$2m + n + 1$

We see that the only case in which $\text{Coker}(\varphi_5)$ does not destabilize \mathcal{F} is the case $(i, j) = (-2, -2)$. Thus, $\text{Ker}(\varphi_1) \simeq \mathcal{O}(-2, -2)$. The cokernel of φ_1 has no zero-dimensional torsion and has Hilbert polynomial $m + 1$, hence it is of the form \mathcal{O}_L for a line L of bidegree $(0, 1)$. From sequence (8) we see that \mathcal{F} is an extension of \mathcal{O}_L by \mathcal{O}_Q .

Conversely, assume that \mathcal{F} is such an extension. By Proposition (3.2) \mathcal{O}_Q is stable. Thus, for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $p(\mathcal{E} \cap \mathcal{O}_Q) < 0$ unless $\mathcal{O}_Q \subset \mathcal{E}$. Since, obviously, \mathcal{O}_L is stable, the image of \mathcal{E} in \mathcal{O}_L has slope at most 1. It follows that $p(\mathcal{E}) < p(\mathcal{F})$, hence \mathcal{F} is stable. From the short exact sequence

$$0 \longrightarrow \mathcal{O}(0, -1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 = \text{Hom}(\mathcal{O}_L, \mathcal{O}_Q) &\longrightarrow H^0(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow H^0(\mathcal{O}_Q(0, 1)) \simeq \mathbb{C}^2 \\ &\longrightarrow \text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \longrightarrow H^1(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow H^1(\mathcal{O}_Q(0, 1)) = 0. \end{aligned}$$

This proves that $\text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$. \square

Let $\mathbf{M}_2 \subset \mathbf{M}$ be the subset of sheaves having resolution (??). Clearly, $\mathbf{M}_2 \simeq \mathbb{P}^{11}$. Let $\mathbf{M}_3 \subset \mathbf{M}$ be the subset of extension sheaves \mathcal{F} as in (15). Clearly, \mathbf{M}_3 is a bundle with base $\mathbb{P}^8 \times \mathbb{P}^1$ and fiber \mathbb{P}^1 . Thus, \mathbf{M}_3 is closed of codimension 3. It intersects \mathbf{M}_2 along a subvariety isomorphic to $\mathbb{P}^8 \times \mathbb{P}^1$ consisting of twisted structure sheaves $\mathcal{O}_C(0, 1)$, where $C = Q \cup L$. The subvarieties $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2 \cup \mathbf{M}_3$ form a decomposition of \mathbf{M} and satisfy the properties from Theorem 1.1.

5. VARIATION OF MODULI OF α -SEMI-STABLE PAIRS

Let X be a separated scheme of finite type over \mathbb{C} . An *algebraic system* on X is a triple $\Lambda = (\Gamma, \sigma, \mathcal{F})$ consisting of an \mathcal{O}_X -module \mathcal{F} , a vector space Γ over \mathbb{C} , and a \mathbb{C} -linear map $\sigma: \Gamma \rightarrow H^0(\mathcal{F})$. If \mathcal{F} is a coherent \mathcal{O}_X -module and Γ is finite dimensional, we say that Λ is a *coherent system*. A *pair* will be a coherent system in which σ is injective and $\dim \Gamma = 1$. A morphism of algebraic systems $(\gamma, \varphi): (\Gamma, \sigma, \mathcal{F}) \rightarrow (\Gamma', \sigma', \mathcal{F}')$ consists of a \mathbb{C} -linear map $\gamma: \Gamma \rightarrow \Gamma'$ together with

a morphism of \mathcal{O}_X -modules $\varphi: \mathcal{F} \rightarrow \mathcal{F}'$, which are compatible, in the sense that $H^0(\varphi)\sigma = \sigma'\gamma$. These notions were introduced in [12] and [10] where appropriate semi-stability conditions of coherent systems were defined, which led in a natural manner to the construction of moduli spaces. The category of algebraic systems on X is abelian and, according to [10, Théorème 1.3], it has enough injectives. Thus, we can define the left derived functors of $\text{Hom}(\Lambda, -)$, denoted $\text{Ext}^i(\Lambda, -)$. Our basic tool for computing these extension spaces is [10, Corollaire 1.6], which we quote below.

Proposition 5.1. *Let $\Lambda = (\Gamma, \sigma, \mathcal{F})$ and $\Lambda' = (\Gamma', \sigma', \mathcal{F}')$ be two algebraic systems on X with σ' injective. Then there is a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(\Lambda, \Lambda') \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{F}') \longrightarrow \text{Hom}(\Gamma, H^0(\mathcal{F}')/\Gamma') \\ &\longrightarrow \text{Ext}^1(\Lambda, \Lambda') \longrightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}') \longrightarrow \text{Hom}(\Gamma, H^1(\mathcal{F}')) \\ &\longrightarrow \text{Ext}^2(\Lambda, \Lambda') \longrightarrow \text{Ext}^2(\mathcal{F}, \mathcal{F}') \longrightarrow \text{Hom}(\Gamma, H^2(\mathcal{F}')). \end{aligned}$$

From now on we specialize to the case when $X = \mathbb{P}^1 \times \mathbb{P}^1$ with fixed polarization $\mathcal{O}(1, 1)$, and \mathcal{F} has dimension 1 with Hilbert polynomial $P_{\mathcal{F}}(m, n) = rm + sn + t$. Let α be a positive rational number. We define the *slope* of a coherent system $\Lambda = (\Gamma, \sigma, \mathcal{F})$ relative to α and to the fixed polarization

$$p_{\alpha}(\Lambda) = \frac{\dim \Gamma}{r+s} \alpha + \frac{t}{r+s}.$$

We say that Λ is α -*semi-stable* (respectively α -*stable*) if \mathcal{F} has no zero-dimensional torsion, σ is injective, and for any proper coherent subsystem $\Lambda' \subset \Lambda$ we have $p_{\alpha}(\Lambda') \leq p_{\alpha}(\Lambda)$ (respectively $p_{\alpha}(\Lambda') < p_{\alpha}(\Lambda)$). According to [10], for fixed polynomial P and $\alpha \in \mathbb{Q}_{>0}$ there is a coarse moduli space $\text{Syst}_{X, \alpha}(P)$ parametrizing S-equivalence classes of α -semi-stable coherent systems (Γ, \mathcal{F}) such that $P_{\mathcal{F}} = P$. We have a decomposition of $\text{Syst}_{X, \alpha}(P)$ into disjoint components according to $\dim \Gamma$. The component corresponding to the case $\dim \Gamma = 1$, i.e. parametrizing α -semi-stable pairs with fixed Hilbert polynomial P , will be denoted $M^{\alpha}(P)$.

A value α_0 is said to be *regular* relative to P if it is contained in an interval (α_1, α_2) such that the set of α -semi-stable pairs with Hilbert polynomial P remains unchanged as α varies in (α_1, α_2) . If there is no such interval we say that α_0 is a *wall* relative to P . The following proposition is analogous to [3, Lemma 3.1].

Proposition 5.2. *Relative to $P(m, n) = 3m + 2n + 1$ we have only one wall at $\alpha = 4$.*

Proof. According to the proof of [10, Théorème 4.2], α is a wall if and only if there is a strictly α -semi-stable pair $\Lambda = (\Gamma, \mathcal{F})$. There is a pair $\Lambda' = (\Gamma', \mathcal{F}') \neq \Lambda$ which is a subpair of Λ or a quotient pair such that $p_{\alpha}(\Lambda') = p_{\alpha}(\Lambda)$. Write $P_{\mathcal{F}'}(m, n) = rm + sn + t$ with $r \leq 3$, $s \leq 2$. We have the equation

$$(16) \quad \frac{\alpha + t}{r + s} = \frac{\alpha + 1}{5}.$$

Without loss of generality we may assume that Γ' generates \mathcal{F}' away, possibly, from finitely many points. Thus $t \geq r + s - rs$. The case when $r = 3$, $s = 2$ is unfeasible. Assume that $r = 2$, $s = 2$, $t \geq 0$. Equation (16) becomes $\alpha = 4 - 5t$, which has solution $\alpha = 4$ when $t = 0$. For all other choices of r and s we have $t \geq 1$, hence equation (16) has no positive solution. \square

We write $\mathbf{M}^\alpha = \mathbf{M}^\alpha(3m + 2n + 1)$. The moduli spaces \mathbf{M}^α remain unchanged as α varies in the interval $(0, 4)$ and will be denoted \mathbf{M}^{0+} . Likewise, for $\alpha \in (4, \infty)$, \mathbf{M}^α are all equal to a moduli space denoted \mathbf{M}^∞ . These moduli spaces are related by the flipping diagram

$$\begin{array}{ccc} \mathbf{M}^\infty & & \mathbf{M}^{0+} \\ & \searrow \rho_\infty & \swarrow \rho_0 \\ & \mathbf{M}^4 & \end{array}$$

in which the maps ρ_∞ and ρ_0 are induced by the inclusion of sets of α -semi-stable pairs. In particular, ρ_∞ and ρ_0 are birational.

The following proposition is a particular case of [18, Proposition B.8].

Proposition 5.3. *The variety \mathbf{M}^∞ is isomorphic to the flag Hilbert scheme of zero-dimensional subschemes of length 2 contained in curves of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.*

In particular, \mathbf{M}^∞ is a bundle with base $\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2)$ and fiber \mathbb{P}^9 , so it is smooth. This proposition gives another proof for the fact that \mathbf{M} is rational (Corollary 4.5).

Remark 5.4. From the proof of Proposition 5.2, we see that the S-equivalence type of a strictly α -semi-stable pair in \mathbf{M}^4 is of the form $(\Gamma, \mathcal{E}) \oplus (0, \mathcal{O}_L)$, where $(\Gamma, \mathcal{E}) \in \mathbf{M}^{0+}(2m + 2n)$ and $L \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a line of bidegree $(0, 1)$. As in the proof of Proposition 3.5, \mathcal{E} has a subsheaf isomorphic to the structure sheaf of a curve. By semi-stability, the curve must have bidegree $(2, 2)$. We see that $\mathcal{E} \simeq \mathcal{O}_Q$ for a quartic curve $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$. Thus, $\mathbf{M}^{0+}(2m + 2n) \simeq \mathbb{P}^8$.

Let $F^\infty \subset \mathbf{M}^\infty$ and $F^0 \subset \mathbf{M}^{0+}$ be the flipping loci, that is, the inverse images under ρ_∞ , respectively, under ρ_0 of $\mathbf{M}^{0+}(2m + 2n) \times \mathbf{M}(m + 1)$. The fiber of F^∞ over (Λ_1, Λ_2) is $\mathbb{P}(\text{Ext}^1(\Lambda_1, \Lambda_2))$. The fiber of F^0 over (Λ_1, Λ_2) is $\mathbb{P}(\text{Ext}^1(\Lambda_2, \Lambda_1))$.

Remark 5.5. The flipping locus F^∞ is a projective bundle with fiber \mathbb{P}^2 and base $\mathbf{M}^{0+}(2m + 2n) \times \mathbf{M}(m + 1)$. The flipping locus F^0 is a \mathbb{P}^1 -bundle with the same base. Indeed, take $\Lambda_1 = (\Gamma, \mathcal{O}_Q) \in \mathbf{M}^{0+}(2m + 2n)$ and $\Lambda_2 = (0, \mathcal{O}_L) \in \mathbf{M}(m + 1)$. Proposition 5.1 yields the exact sequence

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(\Lambda_1, \Lambda_2) \longrightarrow \text{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \text{Hom}(\Gamma, H^0(\mathcal{O}_L)) \simeq \mathbb{C} \\ &\longrightarrow \text{Ext}^1(\Lambda_1, \Lambda_2) \longrightarrow \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \text{Hom}(\Gamma, H^1(\mathcal{O}_L)) = 0. \end{aligned}$$

Any morphism $\Lambda_1 \rightarrow \Lambda_2$ is zero because $\Gamma = H^0(\mathcal{O}_Q)$ generates \mathcal{O}_Q . If $L \not\subset Q$, then $\text{Hom}(\mathcal{O}_Q, \mathcal{O}_L) = 0$; if $L \subset Q$, then $\text{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}$. From the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow H^0(\mathcal{O}_L) \simeq \mathbb{C} \longrightarrow H^0(\mathcal{O}_L(2, 2)) \simeq \mathbb{C}^3 \\ &\longrightarrow \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow H^1(\mathcal{O}_L) = 0. \end{aligned}$$

Thus, if $L \not\subset Q$, then $\text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}^2$; if $L \subset Q$, then $\text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}^3$. In either case we get $\text{Ext}^1(\Lambda_1, \Lambda_2) \simeq \mathbb{C}^3$.

We will now verify the isomorphism $\text{Ext}^1(\Lambda_2, \Lambda_1) \simeq \mathbb{C}^2$. From Proposition 5.1 we have the exact sequence

$$0 = \text{Hom}(0, H^0(\mathcal{O}_Q)/\Gamma) \rightarrow \text{Ext}^1(\Lambda_2, \Lambda_1) \rightarrow \text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \rightarrow \text{Hom}(0, H^1(\mathcal{O}_Q)) = 0$$

Thus, the middle arrow is an isomorphism. From Proposition 4.8 we know that $\text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$.

Lemma 5.6. *For $\Lambda \in F^0$ we have $\text{Ext}^2(\Lambda, \Lambda) = 0$.*

Proof. We have a non-split exact sequence

$$0 \longrightarrow \Lambda_1 \longrightarrow \Lambda \longrightarrow \Lambda_2 \longrightarrow 0$$

for some $\Lambda_1 = (\Gamma, \mathcal{O}_Q) \in M^{0+}(2m+2n)$ and $\Lambda_2 = (0, \mathcal{O}_L) \in M(m+1)$. It is enough to show that $\text{Ext}^2(\Lambda_i, \Lambda_j) = 0$ for $i, j = 1, 2$. From Proposition 5.1 we have the exact sequence

$$0 = \text{Hom}(\Gamma, H^1(\mathcal{O}_L)) \longrightarrow \text{Ext}^2(\Lambda_1, \Lambda_2) \longrightarrow \text{Ext}^2(\mathcal{O}_Q, \mathcal{O}_L) \simeq \text{Hom}(\mathcal{O}_L, \mathcal{O}_Q \otimes \omega)^*.$$

The group on the right vanishes because \mathcal{O}_L is stable, by Proposition 3.2 $\mathcal{O}_Q \otimes \omega$ is stable and $p(\mathcal{O}_L) > p(\mathcal{O}_Q \otimes \omega)$. Thus, $\text{Ext}^2(\Lambda_1, \Lambda_2) = 0$. The exact sequence

$$0 = \text{Hom}(0, H^1(\mathcal{O}_Q)) \rightarrow \text{Ext}^2(\Lambda_2, \Lambda_1) \longrightarrow \text{Ext}^2(\mathcal{O}_L, \mathcal{O}_Q) \simeq \text{Hom}(\mathcal{O}_Q, \mathcal{O}_L \otimes \omega)^* = 0$$

shows that $\text{Ext}^2(\Lambda_2, \Lambda_1) = 0$. We have the exact sequence

$$\begin{aligned} 0 &= \text{Hom}(\Gamma, H^0(\mathcal{O}_Q)/\Gamma) \\ &\longrightarrow \text{Ext}^1(\Lambda_1, \Lambda_1) \longrightarrow \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_Q) \longrightarrow \text{Hom}(\Gamma, H^1(\mathcal{O}_Q)) \simeq \mathbb{C} \\ &\longrightarrow \text{Ext}^2(\Lambda_1, \Lambda_1) \longrightarrow \text{Ext}^2(\mathcal{O}_Q, \mathcal{O}_Q) \simeq \text{Hom}(\mathcal{O}_Q, \mathcal{O}_Q \otimes \omega)^* = 0. \end{aligned}$$

The space $\text{Ext}^1(\Lambda_1, \Lambda_1)$ is isomorphic to the tangent space of $M^{0+}(2m+2n) \simeq \mathbb{P}^8$ at Λ_1 , so it is isomorphic to \mathbb{C}^8 . From the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(\mathcal{O}_Q, \mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow H^0(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow H^0(\mathcal{O}_Q(2, 2)) \simeq \mathbb{C}^8 \\ &\longrightarrow \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_Q) \longrightarrow H^1(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow H^1(\mathcal{O}_Q(2, 2)) = 0. \end{aligned}$$

Thus $\text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_Q) \simeq \mathbb{C}^9$. We get the vanishing of $\text{Ext}^2(\Lambda_1, \Lambda_1)$. Finally, from the exact sequence

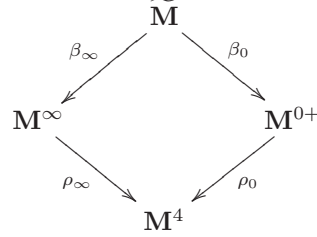
$$0 = \text{Hom}(0, H^1(\mathcal{O}_L)) \longrightarrow \text{Ext}^2(\Lambda_2, \Lambda_2) \longrightarrow \text{Ext}^2(\mathcal{O}_L, \mathcal{O}_L) \simeq \text{Hom}(\mathcal{O}_L, \mathcal{O}_L \otimes \omega)^* = 0$$

we get the vanishing of $\text{Ext}^2(\Lambda_2, \Lambda_2)$. \square

The following theorem is analogous to [3, Theorem 3.3].

Theorem 5.7. *Let M^α be the moduli space of α -semi-stable pairs on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $P(m, n) = 3m + 2n + 1$. We have the following commutative*

diagram expressing the variation of \mathbf{M}^α as α crosses the wall:



Here β_∞ is the blow-up with center F^∞ and β_0 is the blow-down contracting the exceptional divisor \widetilde{F} in the direction of \mathbb{P}^2 , where we regard \widetilde{F} as a $\mathbb{P}^2 \times \mathbb{P}^1$ -bundle over $\mathbf{M}^{0+}(2m+2n) \times \mathbf{M}(m+1)$.

Proof. At [3, Theorem 3.3] a birational map β_0 is constructed from the blow-up $\widetilde{\mathbf{M}}$ of \mathbf{M}^∞ along F^∞ to \mathbf{M}^{0+} , which contracts \widetilde{F} in the \mathbb{P}^2 -directions. Note that β_0 gives an isomorphism on the complement of F^0 and the preimages of points in F^0 are isomorphic to \mathbb{P}^2 . By Remark 5.5, F^0 is smooth. We claim that \mathbf{M}^{0+} is also smooth. This can be verified using the smoothness criterion for moduli spaces of α -semi-stable pairs: if Λ gives a stable point of \mathbf{M}^{0+} and $\text{Ext}^2(\Lambda, \Lambda) = 0$, then Λ gives a smooth point. It is enough to take $\Lambda \in F^0$ and then we can apply Lemma 5.6. We can now apply the Universal Property of the blow-up [9, p. 604], to conclude that β_0 is a blow-up with center F^0 and exceptional divisor \widetilde{F} . \square

The following proposition is analogous to [3, Proposition 4.4]. We define the *forgetful morphism* $\phi: \mathbf{M}^{0+} \rightarrow \mathbf{M}$ by $\phi(\Gamma, \mathcal{F}) = [\mathcal{F}]$.

Proposition 5.8. *The forgetful morphism $\phi: \mathbf{M}^{0+} \rightarrow \mathbf{M}$ is a blow-up of \mathbf{M} along \mathbf{M}_2 .*

Proof. We will give a simpler argument than the one found at [3, Proposition 4.4]. As seen in the proof of Theorem 5.7, \mathbf{M}^{0+} is smooth. The varieties \mathbf{M} and \mathbf{M}_2 are also smooth. Away from \mathbf{M}_2 , ϕ is an isomorphism because, by Theorem 1.1, for $\mathcal{F} \in \mathbf{M} \setminus \mathbf{M}_2$ we have $H^0(\mathcal{F}) \simeq \mathbb{C}$, hence we may identify \mathcal{F} with the α -stable pair $(H^0(\mathcal{F}), \mathcal{F})$ for sufficiently small α . For $\mathcal{F} \in \mathbf{M}_2$, $\phi^{-1}([\mathcal{F}]) = \mathbb{P}(H^0(\mathcal{F})) \simeq \mathbb{P}^1$. By the Universal Property of the blow-up [9, p. 604], ϕ is a blow-up with center \mathbf{M}_2 . \square

Proof of Theorem 1.2. The integral homology groups of \mathbf{M} have no torsion because \mathbf{M}^∞ enjoys this property and \mathbf{M} is obtained from \mathbf{M}^∞ by a sequence of blow-ups and blow-downs. By Theorem 5.7,

$$P(\mathbf{M}^{0+}) = P(\mathbf{M}^\infty) + (P(\mathbb{P}^1) - P(\mathbb{P}^2)) P(\mathbf{M}^{0+}(2m+2n) \times \mathbf{M}(m+1)).$$

By Proposition 5.3 and Remark 5.4,

$$P(\mathbf{M}^{0+}) = P(\mathbb{P}^9) P(\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2)) + (P(\mathbb{P}^1) - P(\mathbb{P}^2)) P(\mathbb{P}^8) P(\mathbb{P}^1).$$

According to [8, Theorem 0.1],

$$P(\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2))(\xi) = \xi^4 + 3\xi^3 + 6\xi^2 + 3\xi + 1.$$

In view of Proposition 5.8,

$$P(\mathbf{M}) = P(\mathbf{M}^{0+}) - \xi P(\mathbf{M}_2) = P(\mathbf{M}^{0+}) - \xi P(\mathbb{P}^{11}).$$

In conclusion,

$$P(\mathbf{M}) = \frac{\xi^{10} - 1}{\xi - 1}(\xi^4 + 3\xi^3 + 6\xi^2 + 3\xi + 1) - \xi^2 \frac{\xi^9 - 1}{\xi - 1}(\xi + 1) - \xi \frac{\xi^{12} - 1}{\xi - 1}.$$

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